Non-Gaussianity from cubic order primordial perturbations: signatures in the CMB

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Plan of talk

- Introduction
  - CMB temperature fluctuations as a random field
  - Initial conditions from inflation
- Simulating CMB maps with non-Gaussian primordial perturbations.
- Measuring non-Gaussianity
  - Genus
- Summary and forthcoming work

CMB observations today

- In any direction in the sky the CMB photons behave like black-body radiation with a temperature T_0.
- WMAP discretizes the sky sphere into about a million pixels and measures the temperature fluctuation in each pixel,

$$\frac{\Delta T(\hat{n})}{T_0} \equiv \frac{T(\hat{n}) - T_0}{T_0}.$$

- $\Delta T(\hat{n})$ gives ONE realization of a 2 dimensional random field with variance of order 10^{-5}.
- In multipoles:

$$\frac{\Delta T(\hat{n})}{T_0} = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}), \quad a_{\ell m} = \int d\Omega \frac{\Delta T(\hat{n})}{T_0} Y^*_{\ell m}(\hat{n}).$$

ΔT as a random field

- Mean:

$$\langle \frac{\Delta T(\hat{n})}{T_0} \rangle = 0, \quad \langle a_{\ell m} \rangle = 0.$$

- Variance:

$$C(\theta) = \langle \frac{\Delta T(\hat{n}_1)}{T_0} \frac{\Delta T(\hat{n}_2)}{T_0} \rangle_{\hat{n}_1 . \hat{n}_2 = \cos\theta}\,, \quad C_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2.$$

If ΔT is Gaussian:

$$C(\theta) = \frac{1}{4\pi} \sum_{\ell} (2\ell+1) C_\ell P_\ell(\theta).$$

- Mean and variance completely characterize the field.
- If ΔT is non-Gaussian we need to know all higher moments to find out its distribution function.

ΔT from theory

$a_{\ell m}$ can be calculated as:

$$a_{\ell m} = 4\pi(-i)^{\ell} \int \frac{d^3 k}{(2\pi)^3} \Phi(\vec{k}, \eta_i) \Delta_{\ell}(k, \eta_0) Y^*_{\ell m}(\hat{k}),$$

- $\Phi(\vec{k}, \eta_i) \equiv$ Fourier transform of primordial metric perturbations $\Phi(\vec{x}, \eta_i)$ at sufficiently early time η_i.
- $\Delta_{\ell}(k, \eta_0)$: radiation transfer function. Determined by the mechanism of creation and the physics around decoupling epoch and subsequent history.

Computing $\Delta_\ell(k, \eta_0)$

- Let f = distribution function for photons.
- At sufficiently early time mean free path of photons $\ll$ Hubble length, and we can assume there is equilibrium in local regions, with T fluctuating from region to region about some average value T_0.
- Then we can write

$$f(\vec{x}, \vec{p}, \eta) = \frac{1}{e^{p/T(\vec{x},\hat{p},\eta)} - 1}, \quad f^{(0)}(p, \eta) \equiv \frac{1}{e^{p/T} - 1},$$

with

$$T(\vec{x}, \hat{p}, \eta) = T_0(1 + \Delta(\vec{x}, \hat{p}, \eta))$$

.

- f can be thought of as perturbed about f^0 as:

$$f(\vec{x}, \vec{p}, \eta) \quad = \quad f^{(0)}(p, \eta) + \delta f(\vec{x}, \vec{p}, \eta),$$

assuming only linear perturbations.

Computing $\Delta_\ell(k,\eta_0)$

$\delta f(\vec{x},\vec{p},\eta)$ can be related to $\Delta(\vec{x},\hat{p},\eta)$ as

$$\delta f(\vec{x},\vec{p},\eta) = -p\frac{\partial f^0}{\partial p}\Delta(\vec{x},\hat{p},\eta).$$

- Time evolution of f can be obtained via Boltzmann equation :

$$\begin{aligned} \frac{df}{dt} &= C[f] \\ C[f] &= \textit{collision term}. \end{aligned}$$

- This translates into a first order equation for $\Delta(\vec{x},\hat{p},\eta)$ with all the interactions going in as the source terms.
- Then we do the following:
  - Fourier transform the $\vec{x}$ dependence.
  - Define $\mu \equiv \hat{k}.\hat{p}$ and transform to multipoles ℓ.
  - Integrate in time.
- The final result is $\Delta_\ell(k,\eta_0)$. Computed by publicly available codes : CMBFAST, CAMB.

Initial conditions $\Phi(\vec{k}, \eta_i)$: inflation

- Inflation tells us $\Phi(\vec{k}, \eta_i)$ is a random field since it comes from vacuum fluctuations of inflaton.
- Variance given by inflationary power spectrum:

$$P_\Phi(k) \sim \langle \Phi_k \Phi_k \rangle = \frac{A_0}{k^3} \left(\frac{k}{k_0} \right)^{n_s - 1} .$$

- Hence by studying the properties of $\Delta T(\hat{n})$ we are 'directly' probing properties of the inflaton field.

How can ΔT be non-Gaussian ?

$a_{\ell m}$ can be non-Gaussian due to :

1. non-linear transfer function $\Delta_\ell(k, \eta_0)$.
   - Expected to be very small since linear perturbation theory has proved to be a very good approximation.
2. Or non-Gaussian $\Phi(k)$:
   - Inflationary perturbation theory when treated to non-linear order predict deviation of $\Phi(k)$ from Gaussianity. True of ALL models. Can express deviation as:

$$\begin{aligned} \Phi &= \Phi^G + \Delta\Phi, \\ \Delta\Phi &\ll \Phi^G. \end{aligned}$$

   - The amount and the functional form of deviation is model dependent. Detailed knowledge can discriminate between different models.

Prediction of $\Delta\Phi$ from inflation in Fourier space

Second order correction to Φ^G is of order $\sim (\Phi^G)^2$. [Salopek & Bond (1990)]

3-point function :

$$\langle \Phi(\vec{k}_1)\Phi(\vec{k}_2)\Phi(\vec{k}_3)\rangle = (2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\ F(k_1,k_2,k_3).$$

- Different models predict different magnitude and shape of $F(k_1,k_2,k_3)$.

4-point function :

$$\langle \Phi(\vec{k}_1)\Phi(\vec{k}_2)\Phi(\vec{k}_3)\Phi(\vec{k}_4)\rangle = (2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4))\ H(k_1,k_2,k_3,k_4).$$

- Different models predict different magnitude and shape of $H(k_1,k_2,k_3,k_4)$.

Prediction of $\Delta\Phi$ from inflation in configuration space

- Schematically

$$\Phi(\vec{x}) = \Phi^G(\vec{x}) + \int d^3y\, d^3z\, K(\vec{y},\vec{z})\, \Phi^G(\vec{x}-\vec{y})\, \Phi^G(\vec{x}-\vec{z}) + \ldots .$$

- Consider the simplified ansatz

$$\Phi(\vec{x}) \quad = \quad \Phi^G(\vec{x}) + f_{NL}\left((\Phi^G(\vec{x}))^2 - \langle(\Phi^G)^2\rangle\right) + \ldots$$

  - Characterized by non-linearity parameter f_{NL}.
  - *Local* since the non-linear contributions depend only on same spatial point.
- f_{NL} is very well studied theoretically and observationally. The tightest constraint from WMAP observation so far

$$-4 < f_{\rm NL} < 80 \quad (95\% CL).$$

Smith and Zaldariagga (2009)

Beyond f_{NL}

We can write $\Phi(\vec{x})$ to third order as:

$$\Phi(\vec{x}) = \Phi^G(\vec{x}) + f_{NL}\left((\Phi^G(\vec{x}))^2 - \langle(\Phi^G)^2\rangle\right) + g_{NL}(\Phi^G(\vec{x}))^3 + \dots$$

- Becomes relevant if g_{NL} can be relatively large.
- Several recent works have shown that in curvaton models or multibrid models or ekpyrotic scenario it can happen that f_{NL} is small or even zero while g_{NL} can be large $\sim \mathcal{O}(10^5)$.

  Allen,Grinstein & Wise (1987)
  Sasaki, Valiviita & Wands (2006), Byrnes, Sasaki & Wands (2006), Enqvist & Takahashi (2008), Huang (2008), PC & Huang (2009); Sasaki (2008), Huang (2009);
  Renaux-Petel (2009).
- $f_{\rm NL}$ can be zero dues to symmetry such as $\Phi \to -\Phi$ or special cancellations of terms.

Simulating non-Gaussian maps

Why is it important :

- it is just solving the time evolution of the temperature fluctuations and hence understanding what theory is predicting.
- more importantly, the simulations can be used as testbeds to study what we should be looking for in the observational data.
- Numerically highly non-trivial, need fast method.

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Simulating non-Gaussian maps

Liguori, Mattarese & Moscardini (2003)
Rewrite $a_{\ell m}$ as real space integral

$$a_{\ell m} = \int dr\, r^2 \Phi_{\ell m}(r) \Delta_\ell(r)$$

where

$$\begin{aligned}
\Delta_\ell(r) &\equiv \frac{2}{\pi} \int dk\, k^2\, \Delta_\ell(k) j_\ell(kr)\ , \\
\Phi_{\ell m}(r) &\equiv \frac{(-i)^\ell}{2\pi^2} \int dk\, k^2\, \Phi_{\ell m}(k)\, j_\ell(kr)\ , \\
\Phi_{\ell m}(k) &= 4\pi (i)^\ell \int dr\, r^2\, \Phi_{\ell m}(r)\, j_\ell(kr)\ ,
\end{aligned}$$

$\Phi_{\ell m}(r)$ split into Gaussian and non-Gaussian parts:

$$\Phi_{\ell m}(r) \equiv \Phi^{G}_{\ell m}(r) + f_{NL} \Phi^{NG}_{\ell m}(r) + g_{NL} \Phi^{NNG}_{\ell m}(r)$$

Simulating non-Gaussian maps

First generate $\Phi^G_{\ell m}(r)$ in (ℓ, m, r) space.

To compute $\Phi^{NG}_{\ell m}(r)$:

- Harmonic transform to get $\Phi^G(\vec{r}) = \sum_{\ell m} \Phi^G_{\ell m}(r) Y_{\ell m}(\hat{r})$.
- Square at each $\vec{r}$, subtract variance to get $\Phi^{NG}(\vec{r})$.
- Inverse harmonic transform to get $\Phi^{NG}_{\ell m}(r) = \int d\Omega \Phi^{NG}_{\ell m}(r) Y^*_{\ell m}(\hat{r})$.

To get $\Phi^{NNG}_{\ell m}(r)$:

- Harmonic transform to get $\Phi^G(\vec{r}) = \sum_{\ell m} \Phi^G_{\ell m}(r) Y_{\ell m}(\hat{r})$.
- Cube at each $\vec{r}$ to get $\Phi^{NG}(\vec{r})$.
- Inverse harmonic transform to get $\Phi^{NNG}_{\ell m}(r) = \int d\Omega \Phi^{NNG}_{\ell m}(r) Y^*_{\ell m}(\hat{r})$.

Inputs for the simulations

Physical parameters of ΛCDM:

- Used WMAP 5-year best fit parameter values.
- Taken $n_s = 1$.
- Set $f_{NL} = 0$.

Resolution:

- Used $\ell_{max} = 1100$.
- Resolves points of angular separation $\theta \sim 9.8\, arcmin$
- Used Healpix $Nside = 512$, corresponds to dividing sphere into about 3×10^6 pixels.

Normalization

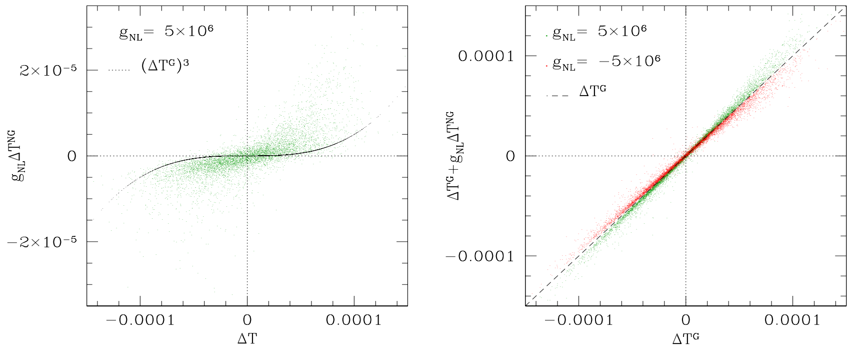
- Gaussian maps normalized by CMBFAST.
- Non-Gaussian maps normalized by matching with Gaussian maps at $\ell = 220$.

Temperature maps

Full temperature fluctuations:

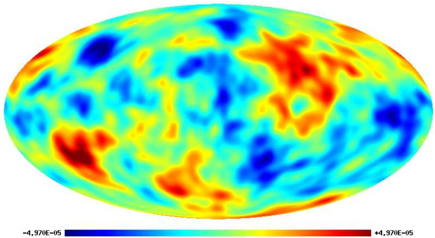
$$\Delta T(\hat{n}) = \Delta T^G + f_{NL} \Delta T^{NG} + g_{NL} \Delta T^{NNG}$$

Distribution of ΔT^{NNG} about ΔT^G:

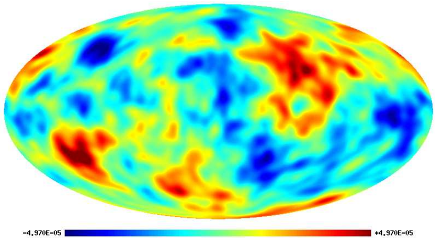


Maps: Positive g_{NL}

Gaussian map with smoothing FWHM$= 7^\circ$:

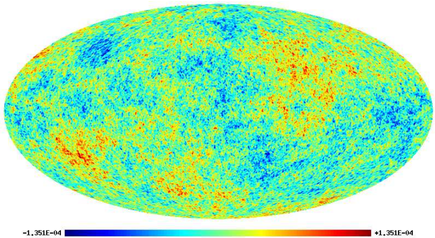


Non-Gaussian map with $g_{NL} = 5 \times 10^6$, same smoothing :

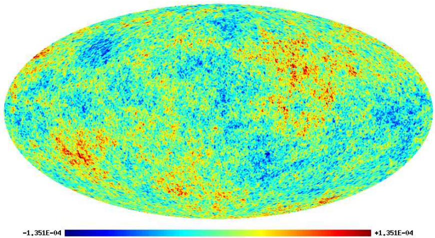


Maps: Positive g_{NL}

Gaussian map with smoothing FWHM$= 30'$:

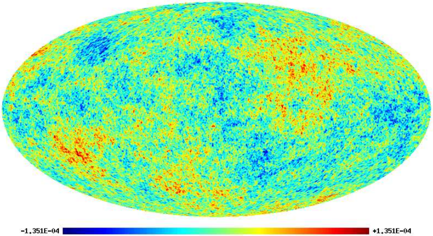


Non-Gaussian map with $g_{NL} = 5 \times 10^6$, same smoothing :

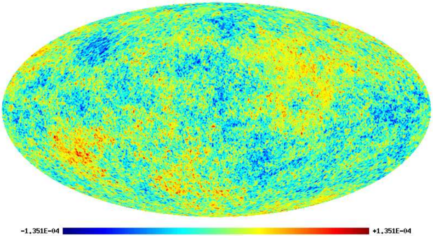


Maps : Negative g_{NL}

Gaussian map with smoothing FWHM$= 30'$:

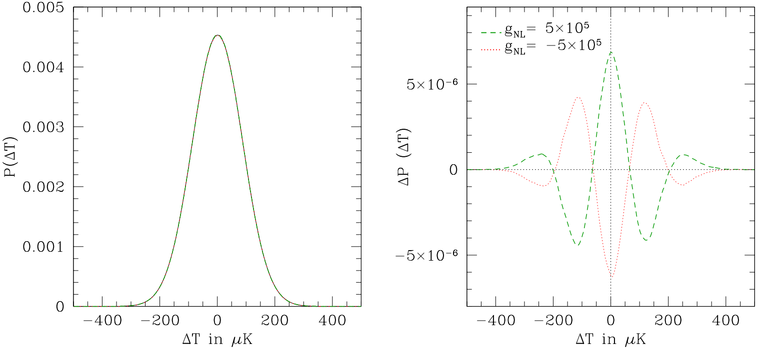


Non-Gaussian map with $g_{NL} = -5 \times 10^6$, same smoothing :



One-point PDF : g_{NL} maps

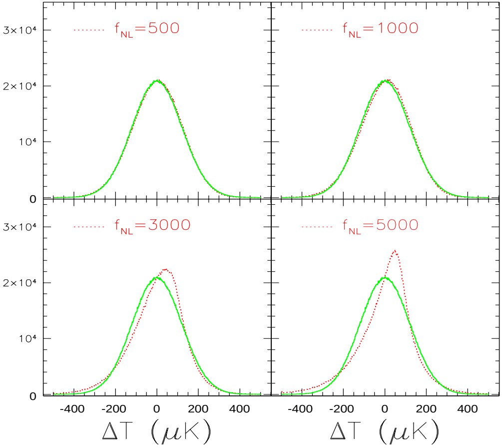
- g_{NL} affects the kurtosis.



One-point PDF : f_{NL} maps

Liguori, Mattarese & Moscardini (2003)

- f_{NL} affects the skewness.



Measuring non-Gaussianity

- Need observables that are sensitive to non-Gaussianity.
- Different statistical tools complement each other and provide cross checks though their sensitivities may vary.
- Defined on harmonic space, pixel (real) space, wavelet space,. . . .

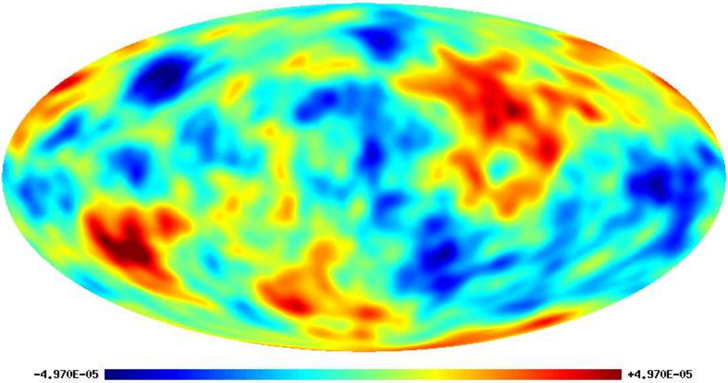
List a few:

1. *Harmonic space observables*: 3-point and 4-point function in multipole space.
2. *Real space observables* : Minkowski functionals, Pixel clustering correlation, etc.
3. *Wavelet, Needlet bispectrum etc.*

Genus: Useful features

- Real space quantity $\Rightarrow$ real world issues such as foreground, galaxy mask etc., are easy to handle.
- Contains information of all correlators since they are global quantities.
- Hence can be more sensitive, compared to measuring individual N-point functions, to other forms of non-Gaussianity than purely f_{NL} or purely g_{NL}.

Genus, G



- $G =$ number of hotspots - number of cold spots.

Genus, G

- Threshold:

$$\nu \equiv \frac{\Delta T/T}{\sigma_0}, \quad \sigma_0 = \sqrt{\langle \frac{\Delta T}{T} \frac{\Delta T}{T} \rangle}.$$

- Then G is given by

$$\begin{aligned} G(\nu) &= \frac{1}{2\pi} \int_C K\, ds \\ C &\equiv contour\ connecting\ pixels\ with\ same\ \nu \\ K &\equiv curvature\ of\ C \end{aligned}$$

Genus for Gaussian field

For Gaussian random field:

$$G = A\,\nu\,e^{-\nu^2}$$

where

$$A = \frac{1}{(2(2\pi)^{3/2}} \frac{\sum \ell(\ell+1)(2\ell+1)\,C_\ell\,W_\ell^2}{\sum(2\ell+1)\,C_\ell\,W_\ell^2}$$

$$W_\ell = e^{-\ell(\ell+1)\theta_s^2/2} = \text{Gaussian smoothing kernel},$$
$$\theta_s = \text{smoothing angle}.$$

It is:

- independent of the normalization.
- depends crucially on the shape of C_ℓ.
- sensitive to the smoothing scale.

Genus for weakly non-Gaussian field

Upto f_{NL} order :

- When the field is *weakly non-Gaussian* approximate analytic expressions may be obtained.
- Then the G's can be expressed as

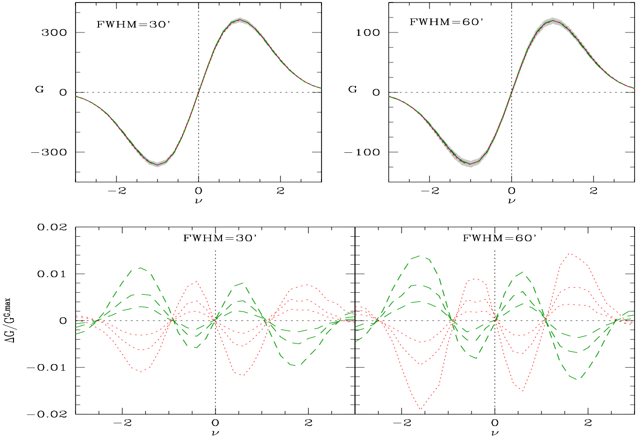
$$G = G^G + \Delta G.$$

- ΔG is known for f_{NL} type non-Gaussianity.
  Matsubara (2003), Hikage et al (2006), Hikage et al (2008).

Upto g_{NL} order :

- No known approximate analytic expressions upto g_{NL}.
- Can be computed directly using simulated non-Gaussian maps.

Genus for g_{NL}

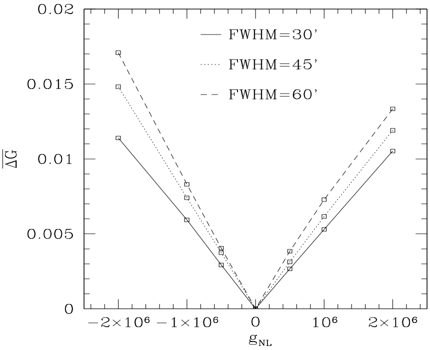


Green represents $g_{NL} > 0$, red $g_{NL} < 0$

Values are $g_{NL} = \pm 5 \times 10^5, \pm 10^6, 2 \times 10^6$

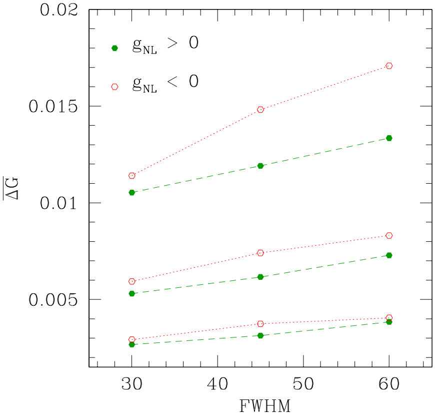
ΔG functional dependence on g_{NL}

- ΔG depends linearly on g_{NL} :



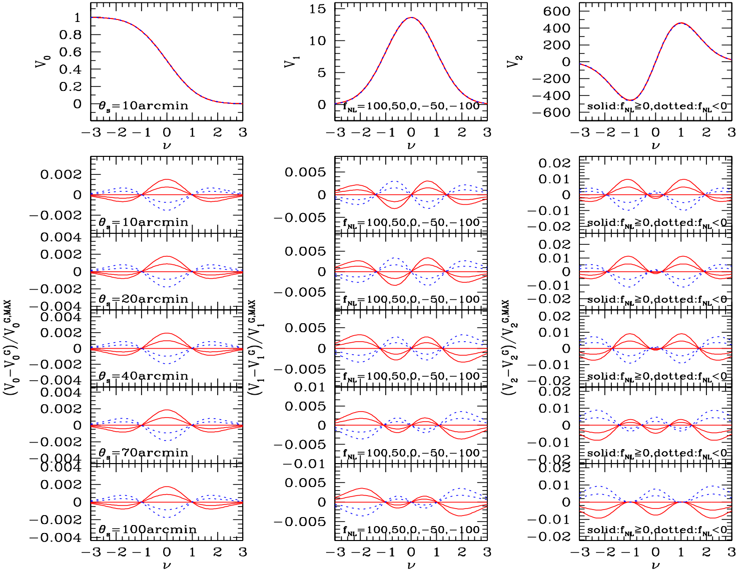
ΔG functional dependence on smoothing scale

- ΔG increases mildly with smoothing scale, at the scales we have probed :



Genus for f_{NL}

Hikage, Komatsu and Matsubara (2006)



Comparing g_{NL} and f_{NL} non-Gaussianities

Characteristic	f_{NL}	g_{NL}
Number of spots	if $f_{NL} > 0$, increases hot spots and decreases cold spots. Vice versa if $f_{NL} < 0$	$g_{NL} > 0$ increases both hot and cold spots. Vice versa if $g_{NL} < 0$
Shape of ΔG	symmetric	anti-symmetric
Dependence on f_{NL} or g_{NL}	linear	linear
Dependence on smoothing	strongly dependent	mildly dependent

Observables derived from genus

- $G(\nu)$ at different ν values are strongly (anti-)correlated.
- Can think of derived observables which will maximize the non-Gaussian deviations and also the difference between f_{NL} and g_{NL}.
- They can then be used to compare with observations to constrain f_{NL} and g_{NL}.

Observables derived from genus

List four observables:

- $$\mathrm{R_{cold}} \equiv \frac{\mathrm{N_{cold}}}{\mathrm{N^G_{cold}}}, \qquad \mathrm{R_{hot}} \equiv \frac{\mathrm{N_{hot}}}{\mathrm{N^G_{hot}}}.$$

$$\mathrm{N_{cold}} \equiv \int_{-\nu_2}^{-\nu_1} d\nu\, G(\nu), \quad \mathrm{N_{hot}} \equiv \int_{\nu_1}^{\nu_2} d\nu\, G(\nu)$$

$$\mathrm{N^G_{cold}} \equiv \int_{-\nu_2}^{-\nu_1} d\nu\, G^{\mathrm{fit}}(\nu), \quad \mathrm{N^G_{hot}} \equiv \int_{\nu_1}^{\nu_2} d\nu\, G^{\mathrm{fit}}(\nu).$$

Gaussian : $\mathrm{R_{cold}} = 1,\ \mathrm{R_{hot}} = 1$
Choose $\nu_1 = 1,\ \nu_2 = 2.5,$

$$\text{If } g_{\mathrm{NL}} > 0,\ \mathrm{R_{cold}} < 1,\ \mathrm{R_{cold}} < 1$$
$$\text{If } g_{\mathrm{NL}} < 0,\ \mathrm{R_{cold}} > 1,\ \mathrm{R_{cold}} > 1$$

Observables derived from genus

- $$\mathrm{R_{spots}} = \frac{\mathrm{N_{cold}} + \mathrm{N_{hot}}}{\mathrm{N^G_{cold}} + \mathrm{N^G_{hot}}}$$

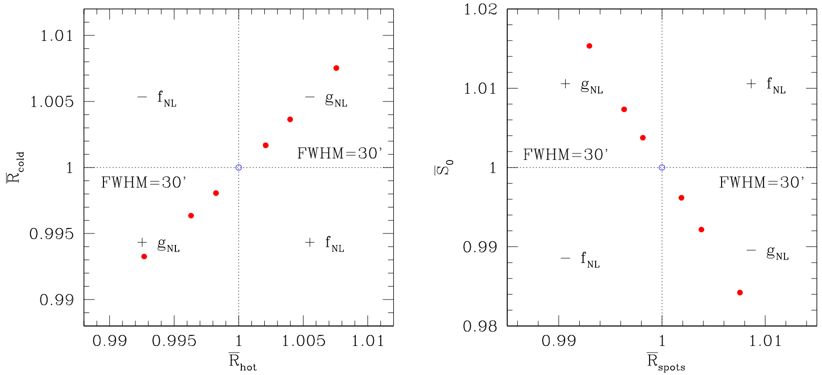
  Gaussian : $\mathrm{R_{spots}} = 1$.

  $$\begin{aligned} \textit{If}\ \ g_{\mathrm{NL}} > 0,\ \mathrm{R_{spots}} &< 1 \\ \textit{If}\ \ g_{\mathrm{NL}} < 0,\ \mathrm{R_{spots}} &> 1 \end{aligned}$$

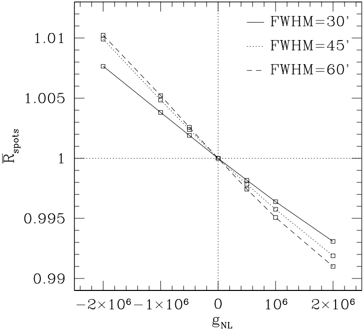
- $\mathrm{S_0}$: ratio of slope of non-Gaussian genus curve to fitted Gaussian at $\nu = 0$.

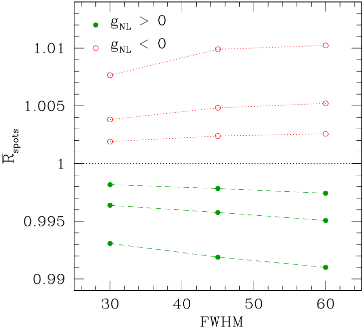
  $$\begin{aligned} \textit{If}\ \ g_{\mathrm{NL}} > 0,\ \mathrm{S_0} &> 1 \\ \textit{If}\ \ g_{\mathrm{NL}} < 0,\ \mathrm{S_0} &< 1 \end{aligned}$$

R_{cold} *versus* R_{hot} and R_{spots} *versus* S_0



Functional dependence of $\overline{R}_{\rm spots}$ on g_{NL} and smoothing scale





Summary

- Simulated non-Gaussian CMB maps arising from primordial perturbations upto cubic order.
- Studied the statistical nature of the non-Gaussian effects on the CMB.
- Measured genus using the simulations and studied how they vary as a function of g_{NL} and smoothing scale.
- We showed f_{NL} and g_{NL} have very different signatures in the genus and other derived observables and can be easily distinguished.
- No observational contaminants such as galaxy mask, point sources, noise, etc were added in this work. Need to include them to compare with real data.
- We are now working on constraining g_{NL} by comparing the simulations with WMAP 5 year data.